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SELF-SIMILAR BREAKUP OF NON-NEWTONIAN LIQUID JETS

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ABSTRACT

We review similarity solutions to describe the asymptotics of the approach to breakup in non-Newtonian liquid jets. Both creeping flow and flow with inertia are considered for a variety of constitutive models. New phenomena in some models of non-Newtonian flows include the possibility of a jet breaking up simultaneously over a finite length, as well as a purely elastic breakup mechanism in which surface tension plays no role.

KEYWORDS: Similarity solutions; Breakup; Liquid jets; Non-Newtonian.

1. INTRODUCTION

Observations of the breakup of liquid jets into droplets go back at least to the ancient Egyptians, who used water dripping from an orifice as a basis for timekeeping devices (as did other cultures independently). The first mention of the effect in the scientific literature is in a book by Mariotte [1]. He assumes, incorrectly, that gravity is the force driving the breakup. In the nineteenth century, systematic experiments to investigate the phenomenon were conducted, most notably by Bidone [2], Savart [3], Hagen [4] and Magnus [5]; these experiments clearly showed that jet breakup resulted from an instability. On the theoretical side, Laplace [6] and Young [7] developed the theory of surface tension, and Plateau [8] noted that a cylinder is not a minimal surface, thereby identifying the mechanism of instability.

The first linear stability analysis, for the surface tension driven instability of an inviscid cylindrical jet, is due to Rayleigh [9, 10]. The extension of the results to the viscous case was finally completed by Chandrasekhar [11], following partial results by Rayleigh [12] and Weber [13]. Tomotika [14] first considered the effect of another liquid outside the jet.

Experiments on jets of viscoelastic fluids (Goldin et al [15], Gordon et al [16]) showed that breakup is significantly delayed compared to the Newtonian case or even suppressed altogether. Linear stability has nothing to do with this; actually viscoelastic jets are more unstable than Newtonian jets with the same viscosity (Goldin et al [15], Kroesser and Middleman [17], Middleman [18]). These results led to the conclusion

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that the stabilizing effect of viscoelasticity arises not in the initial growth of disturbances but in the later stages of jet deformation when the high elongational resistance of the polymer becomes important (Goldin [15], Entov [19]). Numerical simulations (Entov and Yarin [20], Bousfield et al [21]) confirm this. The stabilization of liquid jets by polymeric additives was exploited by spiders long before anybody heard of the Egyptians and their water clocks.

In this article, we shall be concerned with the asymptotic evolution of liquid jets as breakup is approached. In this limit, the evolution of the jet can be described by a similarity solution of a one-dimensional equation that is based on a slender body approximation. Even in the Newtonian case, these similarity solutions were found quite recently. Eggers [22] (see also Eggers [23], Brenner et al [24]) found a similarity solution, which balances viscous, inertial and surface tension forces against each other and leads to breakup of the jet in finite time. This solution was compared with numerical simulations (Eggers and Dupont [25]). Papageorgiou [26] found a different similarity solution for the case in which inertia is neglected. An *a posteriori* analysis shows, however, that in the limit of breakup the assumption of neglecting inertia becomes inconsistent. For highly viscous fluid jets, one would therefore expect a breakup behavior which first follows the Papageorgiou solution, but then changes to that of Eggers as one gets very close to breakup. This expectation has indeed been verified in experiments (Rothert et al [27]).

I shall not attempt to review the extensive literature on viscoelastic jets and filaments, see for instance the books of Petrie [28] and Yarin [29]. Rather, this article will focus on the narrower issue of similarity solutions for breakup. For the Oldroyd B and Maxwell models, it has been shown that no finite time breakup occurs, at least in a one-dimensional model, which neglects axial curvature and inertia (Renardy [30, 31]). The elastic resistance to stretching in these model fluids is strong enough to suppress breakup entirely. On the other hand, Chang et al. [32] have presented numerical results that show a breakup by iterated stretching; axial curvature and inertia were included in their model. A full mathematical analysis of this breakup remains an open problem. Other models for viscoelastic fluids, and most real fluids, are less elongation thickening than the Oldroyd B fluid; for such fluids similarity solutions analogous to those of Eggers and Papageorgiou are possible. Some fluids even decrease their elongational resistance to such an extent that a purely elastic mechanism for breakup is possible in which surface tension plays no role. Such an elastic breakup was observed in the numerical simulations of Hassager et al [33].

Similarity solutions for breakup without inertia were first found for the Giesekus model (Renardy [34]) and subsequently for a number of other models of elastic fluids (Renardy [35, 36], Fontelos [37], Doshi and Basaran [38]). As in the Newtonian case, inertia will change the breakup asymptotics; since many elastic liquids are highly viscous, however, this will often not happen until very close to breakup. Similarity solutions for breakup with inertia were analyzed in Renardy and Losh [39] for the Giesekus model and in Renardy and Renardy [40] for the generalized Newtonian fluid; the numerical results of Doshi [41] show the evolution towards self-similar behavior from generic initial data. An interesting aspect of the analysis for the generalized Newtonian fluid is the existence of branches of solutions, which connect the inertial Eggers solution for the Newtonian case to the inertialess Papageorgiou solution.

2. SLENDER BODY APPROXIMATION

One-dimensional models for liquids jets are based on the assumption of a slender jet where the scale on which the jet radius varies is sufficiently longer than the jet radius and the variation of axial velocity across the jet is negligible. When a jet breaks up into spherical droplets, these assumptions are valid in the necks between the drops where breakup takes place. For the derivation of one-dimensional models for viscoelastic jets, see e.g. Petrie [28], Renardy [42], Bechtel et al [43]; the derivation given below is basically that presented in [28].

We find it convenient to formulate the one-dimensional equations in a Lagrangian formulation. We consider a reference configuration in which the jet has uniform thickness δ. Let X denote the position of a fluid particle in this reference configuration, and let $x(X,t)$ be the actual position. The stretch is defined by:

$$s(X,t)=\frac{\partial x(X,t)}{\partial X}. \qquad \text{..........(1)}$$

Let $u(X,t)$ denote the axial velocity. The equality of mixed partial derivatives leads to:

$$s_t = u_X . \qquad \text{..........(2)}$$

The cross section of the jet is $A=\pi\delta^2/s$. The balance of axial momentum yields:

$$\pi\delta^2 \rho\, u_t = \frac{\partial}{\partial X}\left[A\left(T_{xx}-p\right)+2\pi T\frac{\delta}{\sqrt{s}}\right]. \qquad \text{..........(3)}$$

Here T_{xx} denotes the axial stress component, p is the pressure, and $2\pi T\delta/\sqrt{s}$ is the product of the surface tension coefficient T with the circumference, i.e. the axial force of surface tension. The free surface condition on the lateral free surface of the jet implies that:

$$T_{rr}-p=-\frac{T\sqrt{s}}{\delta}. \qquad \text{..........(4)}$$

By inserting the resulting expression for p in equation (3), we obtain:

$$\rho\, u_t = \frac{\partial}{\partial X}\left(\frac{T_{xx}-T_{rr}}{s}+\frac{T}{\delta\sqrt{s}}\right). \qquad \text{..........(5)}$$

To supplement the equations (5) and (2), we need to relate the stresses to the motion. For the slender body approximation, it is assumed that the axial velocity u is uniform across the cross section of the jet and that the radial velocity is proportional to the radius to satisfy the incompressibility condition. Moreover, we note that:

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}=\frac{s_t}{s}, \qquad \text{..........(6)}$$

according to equation (2). This leads to the velocity gradient:

$$\begin{pmatrix} \frac{s_t}{s} & 0 & 0 \\ 0 & -\frac{s_t}{2s} & 0 \\ 0 & 0 & -\frac{s_t}{2s} \end{pmatrix}. \qquad \text{..........(7)}$$

For the Newtonian fluid, we therefore obtain:

$$T_{xx} = 2\eta \frac{s_t}{s} \ , \ T_{rr} = -\eta \frac{s_t}{s} . \qquad \text{..........(8)}$$

In the following, we shall be interested in similarity solutions for breakup. With the breakup time set at $t = 0$, such solutions have the form:

$$s(X,t) = (-t)^{-\alpha} \phi\left(\frac{X}{(-t)^{\beta}}\right) \ , \ u(X,t) = (-t)^{\beta-\alpha-1} \psi\left(\frac{X}{(-t)^{\beta}}\right). \qquad \text{..........(9)}$$

Before we examine such solutions in detail, we make some general remarks. First, in the derivation of the model equation, it was assumed that azimuthal curvature is large relative to axial curvature. We can assess the consistency of this approximation as breakup is approached. The azimuthal curvature is proportional to $s^{1/2}$, i.e. $(-t)^{-\alpha/2}$. On the other hand, axial curvature is proportional to:

$$s^{-2}\left(s^{-1/2}\right)_{XX} \sim (-t)^{5\alpha/2-2\beta} . \qquad \text{..........(10)}$$

The assumption of neglecting axial curvature is therefore consistent if $\beta < 3\alpha/2$. In much of the analysis below, inertia will be neglected. For this to be consistent, the Reynolds stresses must remain small compared to the elastic or viscous stresses. The Reynolds stress is proportional to u^2, i.e. to $(-t)^{2\beta-2\alpha-2}$. If surface tension is driving the breakup, then the stresses are of the same order as the curvature, i.e. proportional to $s^{1/2} \sim (-t)^{-\alpha/2}$. It follows that the assumption of neglecting inertia is consistent if $\beta > 3\alpha/4+1$. This is violated in most cases, including the Newtonian fluid. For sufficiently viscous fluids, inertialess solutions can describe the evolution up to a point very close to breakup, but ultimately inertia must enter the balance.

The position of a fluid particle in physical space is:

$$x = \int_0^X s(Y,t)\, dY \sim (-t)^{\beta-\alpha} . \qquad \text{..........(11)}$$

Solutions discussed below have either $\beta > \alpha$ or $\beta = \alpha$. In the former case, the self-similar region shrinks to a point as breakup is approached. If $\beta = \alpha$, on the other hand, the self-similar region occupies a fixed length in space and, at breakup, the jet breaks over a finite length at once rather than just at one point. This behavior has been observed in experiments (G. McKinley, private communication).

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3. SIMILARITY SOLUTIONS WITHOUT INERTIA: THE NEWTONIAN CASE

For a Newtonian fluid with no inertia, equation (5) reduces to:

$$\frac{\partial}{\partial X}\left(3\eta\frac{s_t}{s}+\frac{\sigma}{\delta\sqrt{s}}\right)=0\,. \qquad \text{..........(12)}$$

The only physical constants involved are the viscosity η and the surface tension coefficient σ. These constants can be used to form a velocity, but not a length or time scale. A result of this is that the equations have a scaling invariance, and similarity solutions always exist in one-parameter families. We can rescale t with the factor $3\eta\delta/\sigma$, X with δ and u with $\sigma/(3\eta)$. This leads to the dimensionless system:

$$\frac{\partial}{\partial X}\left(\frac{s_t}{s}+\frac{1}{\sqrt{s}}\right)=0\ ,\ s_t=u_X\,. \qquad \text{..........(13)}$$

resulting from equations (12) and (2). We note that if $s(X,t)$, $u(X,t)$ is a solution, then so is $s(\mu X,t)$, $\mu^{-1}u(\mu X,t)$ for any $\mu > 0$. This invariance under stretching of the spatial variable also applies to non-Newtonian fluids as long as inertia is neglected.

An integration of equation (13) leads to:

$$s_t=\lambda(t)s^2-s^{3/2}\ ,\ s_t=u_X\,. \qquad \text{..........(14)}$$

We shall look for solutions of the form:

$$s(X,t)=t^{-2}\phi\left(\frac{X}{(-t)^{\beta}}\right)\ ,\ u(X,t)=(-t)^{\beta-3}\psi\left(\frac{X}{(-t)^{\beta}}\right)\ ,\ \lambda(t)=-k\,t \qquad \text{..........(15)}$$

Here, $t = 0$ is the breakup time. The second equation of (14) then becomes:

$$2\phi(\xi)+\beta\xi\phi'(\xi)=\psi'(\xi)\,. \qquad \text{..........(16)}$$

where $\xi = X/(-t)^{\beta}$. We assume that the region where the velocity tends to infinity as $t \to 0$ is localized; for this we need to require $\psi(-\infty) = \psi(\infty) = 0$. Consequently it follows that:

$$\int_{-\infty}^{\infty}\left[\,2\phi(\xi)+\beta\xi\phi'(\xi)\,\right]d\xi=0\,. \qquad \text{..........(17)}$$

Inserting equation (15) into the first equation of (14), we find:

$$2\phi(\xi)+\beta\xi\phi'(\xi)=k\phi(\xi)^2-\phi(\xi)^{3/2}\,. \qquad \text{..........(18)}$$

We shall assume regular behavior of the solution at $\xi = 0$:

$$\phi(\xi)=y_0-y_2\xi^2+O\left(\xi^4\right). \qquad \text{..........(19)}$$

This imposes the conditions:

$$2y_0 = ky_0^2 - y_0^{3/2}, \qquad \text{..........(20)}$$

and

$$2 + 2\beta = 2ky_0 - \frac{3}{2} y_0^{1/2}. \qquad \text{..........(21)}$$

This leads to:

$$y_0 = 16(\beta - 1)^2 \;,\; k = \frac{-1 + 2\beta}{8(\beta - 1)^2} \qquad \text{..........(22)}$$

We next make the substitution:

$$\phi(\xi) = \frac{y_0}{\left\{1 + 2u(\xi)^2\right\}^2} \qquad \text{..........(23)}$$

We can then find the solution of the differential equation (18) in the implicit form:

$$\chi(u) := u\left(\beta + u^2\right)^{\beta - 1/2} = C\xi, \qquad \text{..........(24)}$$

Since changing the constant only amounts to a rescaling of ξ (see the remarks on scaling above), we can assume $C = 1$ without loss of generality.

The constraint (17) is, by equation (18), equivalent to:

$$\int_{-\infty}^{\infty} \left[k\phi(\xi)^2 - \phi(\xi)^{3/2} \right] d\xi = 0. \qquad \text{..........(25)}$$

which takes the form:

$$\frac{-1 + 2\beta}{2(\beta - 1)} = \frac{\int_0^{\infty} \chi'(u)\left(1 + 2u^2\right)^{-3} du}{\int_0^{\infty} \chi'(u)\left(1 + 2u^2\right)^{-4} du}. \qquad \text{..........(26)}$$

The integrals can be evaluated (see Gradshteyn and Ryzhik [44], p.299, # 3.259,3), resulting in the equation:

$$\begin{aligned} &2(\beta - 1)\left(\frac{7}{2} - \beta\right) {}_2F_1\left(2, \frac{1}{2}, \frac{7}{2} - \beta, 1 - 2\beta\right) = \\ &\quad (2\beta - 1)(3 - \beta)\ {}_2F_1\left(3, \frac{1}{2}, \frac{9}{2} - \beta, 1 - 2\beta\right). \end{aligned} \qquad \text{..........(27)}$$

The smallest positive root of this equation is at β=2.17487.

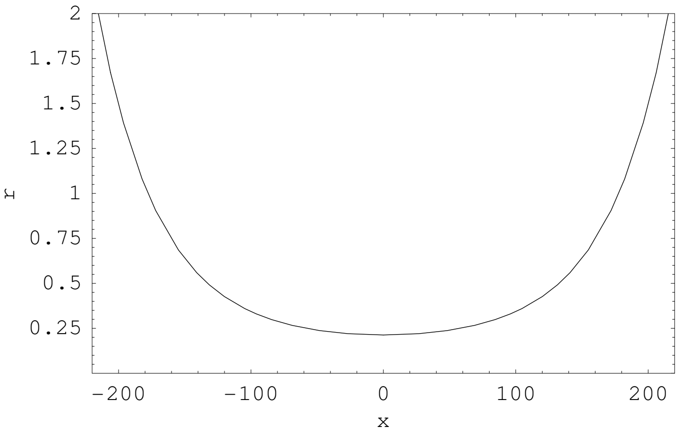

Figure 1: Profile of Papageorgiou's solution.

This solution was first found by Papageorgiou [26] and simplified by Eggers [23]. It has been compared to simulations and experiments and found to describe the breakup from generic initial data. Figure 1 shows the jet profile obtained from the similarity solution. To plot this profile, we have reverted to Eulerian coordinates to show the actual shape in physical space; the plot is of the radius $r = 1/\sqrt{\phi}$ against the position in space:

$$x = \int_0^{\xi} \phi(\zeta)\, d\zeta \,. \qquad \text{..........(28)}$$

Papageorgiou's solution is actually the first in a one-parameter family of solutions. Instead of assuming quadratic behavior at the origin as in equation (19), we can assume $\phi(\xi) \sim y_0 - y_{2n}\xi^{2n}$, where n is any positive integer, and for each n there is a similarity solution [24]. Which similarity solution describes the breakup asymptotics depends on the behavior of the initial data near the point where s has its maximum.

Since $\beta < 5/2 = 1+3\alpha/4$, the assumption of neglecting inertia becomes inconsistent as $t \to 0$. The balance of inertia, viscosity and surface tension leads to another similarity solution for which $\beta = 5/2$. This case will be discussed later.

Lister and Stone [45] show that when the viscosity of an outer liquid is taken into account, then a similarity solution for breakup exists for which the assumption of neglecting inertia is consistent. In this case, the breakup can no longer be modeled by a one-dimensional system and a two-dimensional problem needs to be solved. To my knowledge, no attempts exist at this point to extend such an analysis to non-Newtonian fluids.

4. BREAKUP WITHOUT INERTIA FOR NON-NEWTONIAN FLUIDS

4.1 Generalized Newtonian fluid

For a power law fluid, we have:

$$T_{11} = 2\eta \frac{s_t}{s}\left|\frac{s_t}{s}\right|^{a-1} \quad , \; T_{rr} = -\eta \frac{s_t}{s}\left|\frac{s_t}{s}\right|^{a-1} \quad , \qquad \text{..........(29)}$$

with some exponent $a > 0$. The analogue of equation (12) is, after integration:

$$3\eta \frac{s_t}{s}\left|\frac{s_t}{s}\right|^{a-1} + \frac{\sigma}{\delta} s^{1/2} = \lambda(t) s \, , \qquad \text{..........(30)}$$

and, as before, we can non-dimensionalize and set $3\eta = 1$, $\sigma/\delta = 1$ without loss of generality.

We seek self-similar solutions of the form:

$$s(X,t) = (-t)^{-\alpha} \phi\left(\frac{X}{(-t)^{\beta}}\right) \, , \; \lambda(t) = k(-t)^{\gamma} . \qquad \text{..........(31)}$$

This, together with:

$$\xi = \frac{X}{(-t)^{\beta}} \quad , \qquad \text{..........(32)}$$

transforms equation (30) to:

$$(-t)^{-a}(\alpha\phi + \beta\xi\phi')\left|\,\alpha\phi + \beta\xi\phi'\right|^{a-1} + (-t)^{-\alpha/2}\phi^{a+1/2} - k(-t)^{\gamma-\alpha}\phi^{a+1} = 0 \, . \qquad \text{..........(33)}$$

By matching terms in this equation, we find $\alpha = 2a$, $\gamma = a$, and:

$$(\alpha\phi + \beta\xi\phi')\left|\,\alpha\phi + \beta\xi\phi'\right|^{a-1} + \phi^{a+1/2} - k\phi^{a+1} = 0 \, . \qquad \text{..........(34)}$$

The requirement that:

$$\int_{-\infty}^{\infty} s_t \, dX = 0 \, , \qquad \text{..........(35)}$$

takes the form:

$$\int_{-\infty}^{\infty} \left[2a\phi(\xi) + \beta\xi\phi'(\xi)\right] d\xi = 0 \, . \qquad \text{..........(36)}$$

We assume that near $\xi = 0$:

$$\phi(\xi) = y_0 - y_2\xi^2 + \ldots \quad , \qquad \text{..........(37)}$$

which determines β and k in terms of y_0:

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$$k = \frac{(2a)^a}{y_0} + \frac{1}{\sqrt{y_0}} \ , \ \beta = 1 + 2^{-1-a} y_0^{1/2} a^{-a} \ . \qquad \text{..........(38)}$$

Bounds on β are obtained as follows. The first term in equation (36) is integrated by parts, noting that ϕ is a symmetric function of ξ and $\phi \to 0$ as $\xi \to \pm\infty$:

$$\int_{-\infty}^{\infty} 2a\phi(\xi)\,d\xi = \lim_{M\to\infty}\left[4aM\phi(M) - 2a\int_{-M}^{M} \xi\phi'(\xi)\,d\xi\right]. \qquad \text{..........(39)}$$

Equation (36) becomes:

$$0 = (\beta - 2a)\int_{-\infty}^{\infty} \xi\phi'(\xi)\,d\xi + 4a \lim_{M\to\infty} M\phi(M). \qquad \text{..........(40)}$$

Both terms on the right hand side are positive unless $\beta > 2a$. Moreover, using the asymptotics of the solution at infinity, it is shown in Renardy [35] that the integral in equation (36) diverges to $-\infty$ as $\beta \to 2a+1$. We conclude that $2a < \beta < 2a+1$.

Unlike the Newtonian case, there is no closed form solution available. Numerical solutions were found in Doshi and Basaran [38] and in Renardy and Renardy [40]. For a fixed a, the numerical solution of equations (34) and (36) proceeds as follows: For any positive y_0, we can numerically solve (34), and the integral term in (36) is finite as long as $\beta < 2a+1$. If y_0 is sufficiently small, the integral term is positive, while it becomes negative when β is close to $2a+1$ (Renardy [35]). We can thus find two values of y_0, one that makes the integral in (36) positive and one that makes it negative. To the pair of y_0 values, a bisection method is applied, to find the value of y_0, for which the integral is equal to zero. Finally, equation (38) gives β as a function of y_0 and a.

Figure 2 shows that β is close to $2a+1$ for small a, and close to $2a$ for large a. We note that for $\beta < 3a/2+1$, inertia ultimately becomes important as breakup is approached. Moreover, if $\beta > 3a$, then axial curvature is not negligible close to breakup (Renardy [36]). These lines are drawn dashed in figure 2; the bold dots denote intersections and show that inertia can be neglected only if a is either less than about 0.26 or larger than approximately 1.95; *i.e.*, for fluids that are either strongly shear thinning or strongly shear thickening. Axial curvature is important if a is less than approximately 0.54.

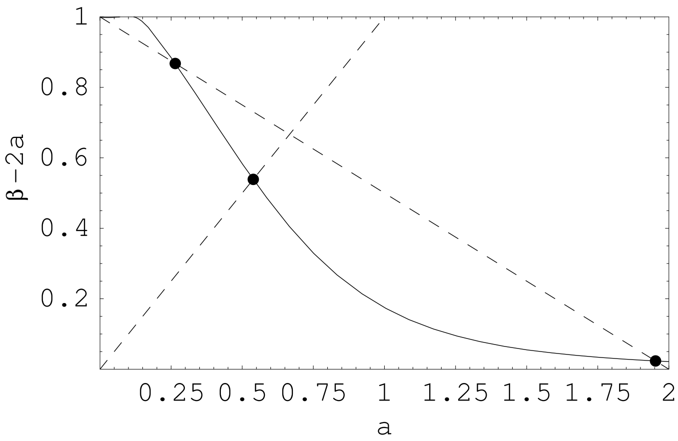

Figure 2: Power law fluid. $\beta - 2a$ as a function of a.

For large ξ , ϕ is proportional to $\xi^{-2a/\beta}$. This implies that the asymptotic behavior of the jet profile in physical space is:

$$r \sim x^{a/(\beta - 2a)}. \qquad \text{..........(41)}$$

This leads to a cusped shape when a is small, and to a very flat U shape when a is large and β is close to $2a$. Figures 3 and 4 illustrate this change.

4.2 Giesekus/PTT model

For these models, we have:

$$T_{11} = 2\eta \frac{s_t}{s} + t_{11} \;,\; T_{rr} = -\eta \frac{s_t}{s} + t_{rr} \quad , \qquad \text{..........(42)}$$

where:

$$\begin{aligned} (t_{11})_t - \frac{2t_{11}s_t}{s} + \kappa\, t_{11} + \nu\, t_{11}^2 &= \frac{2\mu\, s_t}{s} \;, \\ (t_{rr})_t + \frac{t_{rr}s_t}{s} + \kappa\, t_{rr} + \nu\, t_{rr}^2 &= -\frac{\mu\, s_t}{s} \;, \end{aligned} \qquad \text{..........(43)}$$

for the Giesekus fluid, and:

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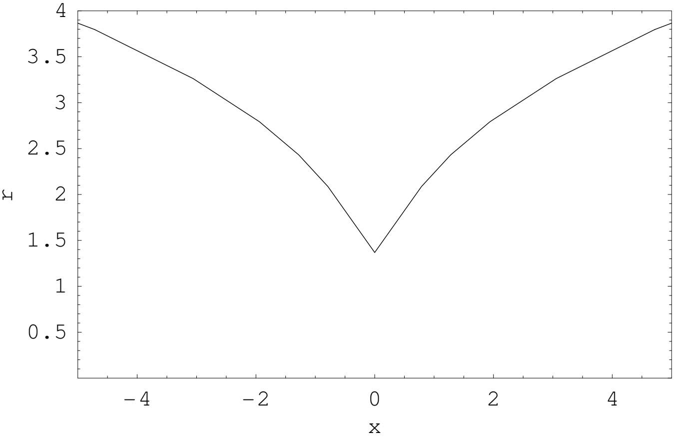

Figure 3: Jet profile for power law liquid, $a = 0.3$

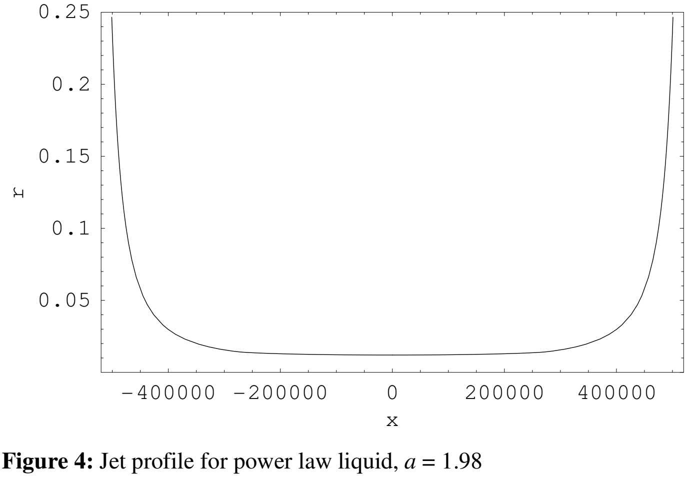

Figure 4: Jet profile for power law liquid, $a = 1.98$

$$\left(t_{11}\right)_t - \frac{2t_{11}s_t}{s} + \kappa\, t_{11} + \nu\left(t_{11} + 2t_{rr}\right)t_{11} = \frac{2\mu\, s_t}{s} \ ,$$
$$\left(t_{rr}\right)_t + \frac{t_{rr}s_t}{s} + \kappa\, t_{rr} + \nu\left(t_{11} + 2t_{rr}\right)t_{rr} = -\frac{\mu\, s_t}{s} \ , \qquad \text{..........(44)}$$

for the PTT fluid. Unlike the Newtonian and power law liquids discussed above, similarity solutions for these models are not exact solutions, but asymptotic solutions in the limit of high stretching, which applies as jet breakup is approached. In this limit, t_{rr} can be neglected and the equation for t_{11} reduces to:

$$\left(t_{11}\right)_t - \frac{2t_{11}s_t}{s} + \nu\, t_{11}^2 = 0 \qquad \text{..........(45)}$$

for both models. We can simplify this equation by setting $t_{11} = ps^2$, leading to the new equation:

$$p_t + \nu\, p^2 s^2 = 0 \qquad \text{..........(46)}$$

for p.

Non-dimensionalization of the equations leads to the single dimensionless parameter [34] $\tilde{\eta} = \eta\nu$, which is a kind of retardation parameter, since $1/\nu$ has the dimension of viscosity. It is, however, not the zero shear rate viscosity, but an elongational viscosity in the limit of infinite elongation rate. In dimensionless form, the equations are:

$$ps^3 + 3\tilde{\eta}\, s_t + s^{3/2} = \lambda(t)s^2 \ ,$$
$$p_t + p^2 s^2 = 0 \ , \qquad \text{..........(47)}$$
$$s_t = u_X \ .$$

For a similarity solution, we set:

$$s(X,t) = t^{-2}\tilde{s}\left(\frac{X}{(-t)^{\beta}}\right), \quad p(X,t) = (-t)^3\,\tilde{p}\left(\frac{X}{(-t)^{\beta}}\right),$$
$$u(X,t) = (-t)^{\beta-3}\,\tilde{u}\left(\frac{X}{(-t)^{\beta}}\right), \qquad \text{..........(48)}$$

and the force $\lambda(t)$ is of the form $\lambda = -kt$. We insert this ansatz into the equations above, and obtain:

$$\tilde{p}\tilde{s}^3 + 3\tilde{\eta}\left[2\tilde{s} + \beta\xi\,\tilde{s}'(\xi)\right] + \frac{\sigma}{\delta}\tilde{s}^{\,3/2} = k\,\tilde{s}^2 \ ,$$
$$-3\tilde{p} + \beta\xi\,\tilde{p}'(\xi) + \nu\,\tilde{p}^2\tilde{s}^2 = 0 \ , \qquad \text{..........(49)}$$
$$2\tilde{s} + \beta\xi\,\tilde{s}'(\xi) = \tilde{u}'(\xi) \ .$$

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We need to satisfy the integral constraint:

$$\int_{-\infty}^{\infty} [\, 2\tilde{s} + \beta\xi\, \tilde{s}'\,]\, d\xi = 0\,. \qquad ....(50)$$

Because of the first equation in (49), we can put this in the alternative form:

$$\int_{-\infty}^{\infty} \left(\tilde{p}\tilde{s}^{3} + \frac{\sigma}{\delta}\tilde{s}^{\,3/2} - k\,\tilde{s}^{2} \right) d\xi = 0 \qquad ....(51)$$

We shall use this constraint to find *k*.

We shall now focus on the case of no retardation, i.e. $\tilde{\eta} = 0$. In this case, the first equation of (49) reduces to:

$$\tilde{p}\tilde{s} + \tilde{s}^{-1/2} = k\ . \qquad ....(52)$$

For a given value of $\tilde{p}$ and $k > 3(2\,\tilde{p}\,)^{1/3}/2$, there are two values of $\tilde{s}$ satisfying equation (52). If we include the $\tilde{\eta}$-term with a small $\tilde{\eta}$ back in the equation, we find that the larger solution of (52) is stable, while the smaller solution is unstable. Moreover, when $\tilde{p} > 4k^3/27$, then $\tilde{s}$ will rapidly approach zero. Hence the appropriate solution to consider for $\tilde{\eta} = 0$ is one where *s* takes the larger value consistent with (52) and then jumps to zero when $\tilde{p}$ reaches the value $4k^3/27$. Let us say this happens at $\xi = \xi_0$. The constraint (50) then reads:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} [\, 2\tilde{s} + \beta\xi\, \tilde{s}'\,]\, d\xi = 2\int_{0}^{\infty} [\, 2\tilde{s} + \beta\xi\, \tilde{s}'\,]\, d\xi \\ &= 2\int_{0}^{\xi_0 +} [\, 2\tilde{s} + \beta\xi\, \tilde{s}'\,]\, d\xi = 2(2-\beta)\int_{0}^{\xi_0} \tilde{s}\, d\xi\ . \end{aligned} \qquad ....(53)$$

Consequently, $\beta = 2$.

To solve the differential equation, we solve equation (52) for $\tilde{p}$ and insert the result in the second equation of (49). We then set $\tilde{s} = \phi^2$. This leads to the differential equation:

$$3\phi - 2k\phi^3 + 6\xi\phi' + \phi^2(1-3k) + k^2\phi^4 - 4k\xi\phi\phi' = 0\,. \qquad ....(54)$$

Near $\xi = 0$, we expect the behavior $\phi = \phi_0 - \phi_2\xi^2 + O(\xi^4)$. This leads to the values $\phi_0 = 6/7$, $k = 21/4$. We next substitute:

$$\phi = \frac{6}{7(1+2u^2)}, \qquad ....(55)$$

which allows us to obtain the solution of the differential equation in the implicit form:

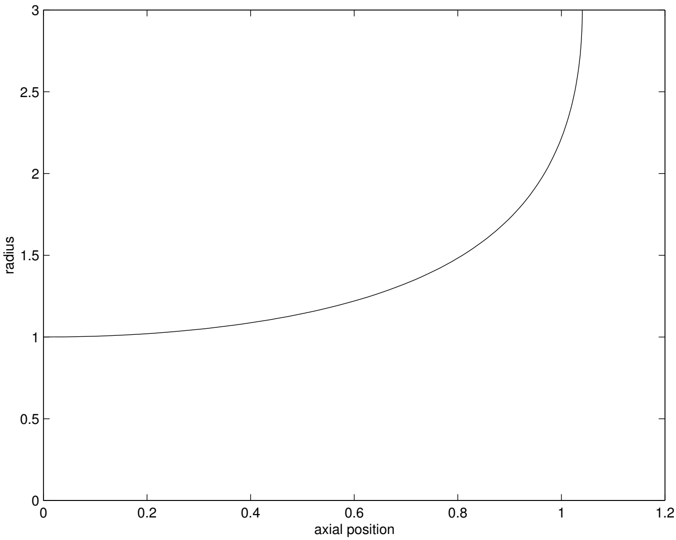

Figure 5: Giesekus jet, profile without retardation.

$$\frac{u\left(7-4u^2\right)^{2/3}}{\left(8+7u^2\right)^{5/6}}=C\xi\,, \qquad \text{..........(56)}$$

By a rescaling of ξ, we can choose $C = 1$. The point where $\tilde{p} = 4k^3/27$ and $\tilde{s}$ jumps to zero corresponds to $u = 1$.

Figure 5 shows the corresponding jet profile. We note that $\beta = 2$ implies that the spatial coordinate $x = \int s\, dX$ does not depend on time. Consequently, the self-similar part of the jet occupies a fixed length in space rather than shrinking to a point as breakup is approached.

Numerical solutions for nonzero $\tilde{\eta}$ were found in Renardy [34]. The value of β varies continuously from $\beta = 2$ for $\tilde{\eta} = 0$ to the Newtonian value of 2.17487 at large $\tilde{\eta}$, see figure 6. The solution $\tilde{s}$ now approaches zero asymptotically as in the Newtonian case rather than jumping to zero at a finite ξ. For small $\tilde{\eta}$, there is an inner region where the solution looks like that for $\tilde{\eta} = 0$ and an outer region where the solution is qualitatively similar to the Newtonian case.

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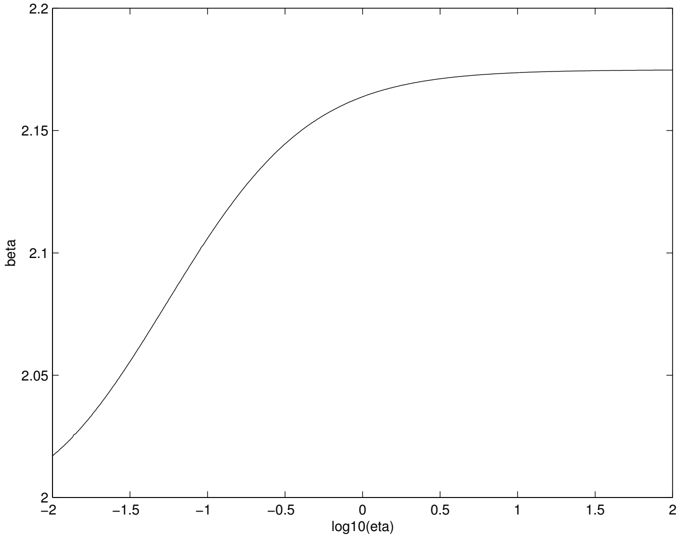

Figure 6: β as a function of $\log_{10}(\tilde{\eta})$.

4.3 Generalized PTT model

We shall now consider a generalized PTT model, for which the quadratic term in the constitutive law in the previous section is replaced by a more general power. That is, we assume the constitutive relation:

$$\frac{d\mathbf{T}}{dt} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T + \kappa\mathbf{T} + \nu(\,tr\mathbf{T})^{a-1}\mathbf{T} = \mu\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right\}. \qquad ....(57)$$

Here $a > 1$. We shall not consider a retardation time, because if it is included, then Newtonian terms would either be negligible or dominate in the limit of jet breakup unless $a = 2$ as in the preceding section.

In the limit of a rapidly stretching jet, we can neglect T_{rr}, and the leading terms in the equation for T_{11} lead to:

$$(T_{11})_t - 2\frac{s_t}{s}T_{11} + \nu\, T_{11}^a = 0\ . \qquad ....(58)$$

As before, we set $T_{11} = ps^2$, and we make the similarity ansatz:

$$s(X,t) = t^{-\alpha}\,\tilde{s}\left(\frac{X}{(-t)^{\beta}}\right), \quad p(X,t) = (-t)^{3\alpha/2}\,\tilde{p}\left(\frac{X}{(-t)^{\beta}}\right),$$
$$u(X,t) = (-t)^{\beta-\alpha-1}\,\tilde{u}\left(\frac{X}{(-t)^{\beta}}\right), \qquad \text{..........(59)}$$

and the force $\lambda(t)$ is of the form $\lambda = k(-t)^{\alpha/2}$. We also non-dimensionalize to remove the constants ν and σ/δ. Balance of terms in equations (5) and (58) leads to:

$$\alpha = \frac{2}{a-1}, \qquad \text{..........(60)}$$

and we find the reduced set of equations:

$$\tilde{p}\tilde{s}^{3} + \tilde{s}^{3/2} = k\,\tilde{s}^{2},$$
$$-\frac{3\alpha}{2}\tilde{p} + \beta\xi\,\tilde{p}'(\xi) + \tilde{p}^{a}\,\tilde{s}^{2a-2} = 0, \qquad \text{..........(61)}$$
$$\alpha\,\tilde{s} + \beta\xi\,\tilde{s}'(\xi) = \tilde{u}'(\xi).$$

We can eliminate $\tilde{p}$ from the first equation of (61) and insert into the second equation. In addition, we substitute $\tilde{s}(\xi) = \psi(\xi)^2$. This results in the equation:

$$2\left(-\psi + k\psi^{2}\right)^{a} + 3\alpha\psi(1-k\psi) + 6\beta\xi\psi' - 4\beta k\xi\psi\psi' = 0. \qquad \text{..........(62)}$$

We seek solutions of equation (62), which have the behavior:

$$\psi(\xi) \sim a_0 + a_2\xi^{2} + O\left(\xi^{4}\right), \qquad \text{..........(63)}$$

near $\xi = 0$. After some calculation, this leads to the conditions:

$$k = \frac{3(-1+2\beta)}{2a_0(-3+2\beta)}, \qquad \text{..........(64)}$$

and

$$\left[\frac{a_0(3+2\beta)}{(4\beta-6)}\right]^{a-1} = \frac{3}{a-1}. \qquad \text{..........(65)}$$

We now look for solutions to equation (62) which start out from a value $a_0 = \psi(0) > 3/(2k)$. With increasing ξ, ψ decreases. According to (62), ψ' then becomes infinite when $\psi = 3/(2k)$. At this point, ψ jumps to zero, as discussed in the previous section for the Giesekus model.

We need to satisfy the integral constraint:

$$\int_{-\infty}^{\infty}\left[\alpha\,\tilde{s} + \beta\xi\,\tilde{s}'\right]d\xi = 0. \qquad \text{..........(66)}$$

If $\tilde{s}$ jumps to zero at a finite ξ_0, this condition reads:

$$0=\int_{-\infty}^{\infty}\left[\alpha\tilde{s}+\beta\xi\tilde{s}'\right]d\xi=2\int_{0}^{\infty}\left[\alpha\tilde{s}+\beta\xi\tilde{s}'\right]d\xi$$
$$=2\int_{0}^{\xi_0+}\left[\alpha\tilde{s}+\beta\xi\tilde{s}'\right]d\xi=2(\alpha-\beta)\int_{0}^{\xi_0}\tilde{s}\,d\xi\ . \qquad ....(67)$$

Consequently, $\beta=\alpha=2/(a-1)$. In equations (64) and (65), this leads to:

$$k=\frac{3(5-a)}{2(7-3a)a_0}\ , \qquad ....(68)$$

and

$$\left[\frac{a_0(1+3a)}{(14-6a)}\right]^{a-1}=\frac{3}{a-1}\ . \qquad ....(69)$$

In order for a_0 to be positive, we need to restrict a to be less than 7/3. In the limit $a\to 7/3$, a_0 tends to zero, and k tends to infinity like $1/a_0^2$. This means that, at the breakup point, the term $k\tilde{s}^{\,2}$ is proportional to a_0^2, while the term $\tilde{s}^{\,3/2}$ (which represents the surface tension force) is proportional to $a_0^2 << a_0^2$. We conclude that in the limit $a\to 7/3$, surface tension ceases to play a role in the breakup.

Motivated by this observation, we shall look for the possibility of breakup without surface tension. We therefore drop the term involving surface tension in the equations of motion. If we do so, there is no longer a reason why $\lambda(t)$ should behave like $(-t)^{\alpha/2}$, and our similarity ansatz becomes:

$$s(X,t)=t^{-\alpha}\,\tilde{s}\left(\frac{X}{(-t)^{\beta}}\right),\quad p(X,t)=(-t)^{\alpha+\gamma}\,\tilde{p}\left(\frac{X}{(-t)^{\beta}}\right),\quad q(X,t)=0,$$
$$u(X,t)=(-t)^{\beta-\alpha-1}\,\tilde{u}\left(\frac{X}{(-t)^{\beta}}\right), \qquad ....(70)$$

and $\lambda(t)=(-t)^{\gamma}$. We obtain the equations:

$$\tilde{p}\tilde{s}=k\,,$$
$$-(\alpha+\gamma)\tilde{p}+\beta\xi\,\tilde{p}'(\xi)+\tilde{p}^{a}\,\tilde{s}^{2a-2}=0\ ,\qquad ....(71)$$
$$\alpha\tilde{s}+\beta\xi\,\tilde{s}'(\xi)=\tilde{u}'(\xi)\ .$$

Moreover, we find the relationship:

$$\gamma=\alpha-\frac{1}{a-1}\ . \qquad ....(72)$$

Combining the first two equations of (71) results in the equation:

$$(k\tilde{s})^{a}-k\left[\left(2\alpha-\frac{1}{a-1}\right)\tilde{s}+\beta\xi\,\tilde{s}'\right]=0 \qquad ....(73)$$

We set $k\tilde{s}=\phi$, and we can solve the differential equation in the form:

$$\phi^{1-a}\left(1+2\alpha-2a\alpha\right)+a-1=C\xi^{(-1-2\alpha+2a\alpha)/\beta} \qquad ....(74)$$

We expect $2a\alpha-2\alpha-1$ to be positive, and ϕ to be a decreasing function of ξ. In this case C must be negative, and we may rescale ξ such that $C = -1$. We obtain quadratic behavior near ξ=0 if:

$$\beta=(a-1)\alpha-\frac{1}{2} \quad . \qquad ....(75)$$

In this case, the solution becomes:

$$\phi(\xi)=\left[\frac{2(a-1)\alpha-1}{a-1+\xi^2}\right]^{1/(a-1)} \quad . \qquad ....(76)$$

Moreover, α is determined by the condition that:

$$\int_0^\infty\left[\,\alpha\phi(\xi)+\beta\xi\,\phi'(\xi)\,\right]d\xi=0\,. \qquad ....(77)$$

This condition leads to:

$$\alpha=\frac{1}{2a-4} \quad . \qquad ....(78)$$

Self-similar solutions for breakup without surface tension therefore exist for $a > 2$. However, if $2 < a < 7/3$, then surface tension, if present, will affect the breakup asymptotics even though breakup is possible without surface tension. For details, see Renardy [35].

The mechanism for breakup without surface tension is a purely elastic one, related to the weakening of the jet when the stretching rate becomes high. We note that jet breakup involves a limit, which combines high stretching rates and large strains. Such a limit can be characterized neither by steady elongational viscosity nor by instantaneous elasticity. Indeed, the generalized PTT fluid discussed here is elongation thinning (in terms of steady elongational viscosity) if $a > 2$, but a generalized Newtonian fluid with the analogous power law does not show breakup without surface tension, see Renardy [35]. On the other hand, the instantaneous elastic behavior of the model considered here is equivalent to that of the Maxwell fluid, which has no breakup at all.

4.4 Other constitutive models

A number of other constitutive models have been investigated for similarity solutions, see Fontelos [37], Renardy [36]. In [36], a non-linear dumbbell model with the Peterlin approximation is considered. For this model, the stress is given by:

$$\mathbf{T}=f\,(\mathrm{tr}\mathbf{C})\,\mathbf{C}\;, \qquad ....(79)$$

where the conformation tensor satisfies the evolution equation:

$$\frac{d\mathbf{C}}{dt} - (\nabla\mathbf{v})\mathbf{C} - \mathbf{C}(\nabla\mathbf{v})^T - \gamma\,\kappa\,\mathbf{I} + \delta\, f(\,tr\,\mathbf{C})\mathbf{C} = \mathbf{0} \;. \qquad \text{..........(80)}$$

If the spring constant *f* is constant, this is equivalent to the upper convected Maxwell model. Another popular choice is the FENE dumbbell, for which c = (*tr* **C**) has a maximum value c_0 and:

$$f(c) \sim \frac{a}{c_0 - c} \quad , \qquad \text{..........(81)}$$

as $c \to c_0$. For this model, the breakup asymptotic is like that of the Newtonian fluid. We can also consider a power law, where:

$$f(c) \sim c^{a-1} \quad , \qquad \text{..........(82)}$$

as $c \to c_0$, where $a > 1$. For this model, an analysis analogous to the generalized PTT model is possible. Similarity solutions were obtained in Renardy [36]. These similarity solutions have $\beta = \alpha = 2a/(a-1)$, and a jump of s to zero at a finite value of ξ, so the self-similar region occupies a fixed finite length. The possibility of breakup without surface tension does not arise.

Fontelos [37] has considered the Johnson-Segalman model with a retardation term:

$$\mathbf{T} = \eta\left[(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\right] + \mathbf{S} \;, \qquad \text{..........(83)}$$

where:

$$\begin{aligned}\frac{d\mathbf{S}}{dt} - \frac{a+1}{2}\left[(\nabla\mathbf{v})\mathbf{S} + \mathbf{S}(\nabla\mathbf{v})^T\right] + \frac{1-a}{2}\left[\mathbf{S}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{S}\right] + \kappa\,\mathbf{T} \\ = \mu\left[(\nabla\mathbf{v})\mathbf{S} + \mathbf{S}(\nabla\mathbf{v})^T\right]\end{aligned} \;. \qquad \text{..........(84)}$$

He finds Newtonian breakup if $a \le 1/4$ and purely elastic breakup with no role of surface tension if $1/4 < a < 1/2$. For a > 1/2 there is no breakup. In [35], it is shown that the same results can be applied to a K-BKZ model if the strain-dependent non-linearity behaves like a power law.

5. SELF-SIMILAR BREAKUP WITH INERTIA

5.1 Newtonian fluid

If inertia is retained in the equations, then equations (5) and (2) take the following form for a Newtonian fluid:

$$\rho\, u_t = \frac{\partial}{\partial X}\left(3\eta\frac{s_t}{s^2} + \frac{\sigma}{\delta\sqrt{s}}\right) \;,\; s_t = u_X \;. \qquad \text{..........(85)}$$

There are now three physical constants, ρ , η and σ , which can be used to form a length scale $l = 9\eta^2/(\rho\sigma)$ and a timescale $\tau = 27\,\eta^3/(\rho\sigma^2)$. Since δ has no intrinsic physical meaning, we can set it equal to l and use these length and time scales for non-dimensionalization. The resulting dimensionless equations are:

$$u_t = \frac{\partial}{\partial X}\left(\frac{s_t}{s^2} + \frac{1}{\sqrt{s}}\right), \ s_t = u_X . \qquad ....(86)$$

For a similarity solution, we assume the form:

$$s(X,t) = (-t)^{-\alpha}\phi\left(\frac{X}{(-t)^{\beta}}\right), \ u(X,t) = (-t)^{\beta-\alpha-1}\psi\left(\frac{X}{(-t)^{\beta}}\right). \qquad ....(87)$$

As before, we set $\xi = X/(-t)^{\beta}$. We note that our origin of coordinates corresponds to Eggers' “stagnation point."

Balance of inertial, viscous and surface tension terms in the momentum equation leads to $\alpha = 2$, $\beta = 5/2$, and to the reduced equations:

$$\frac{1}{2}\psi + \frac{5}{2}\xi\psi' = \frac{d}{d\xi}\frac{2}{\varphi} + \frac{5}{2}\xi\frac{\varphi'}{\varphi^2} + \frac{1}{\sqrt{\varphi}} \ ,$$
$$2\phi + \frac{5}{2}\xi\phi' = \psi' \ . \qquad ....(88)$$

At zero, these equations have a singular point. We look for solutions that are analytic and have the form:

$$\phi(\xi) = s_0 + s_1\xi + s_2\xi^2 + s_3\xi^3 + ...$$
$$\psi(\xi) = u_0 + u_1\xi + u_2\xi^2 + u_3\xi^3 + ... \ . \qquad ....(89)$$

We find that s_0 and u_0 can be prescribed arbitrarily and the coefficients of the higher order terms can then be determined from the equations. The expansion at zero breaks down, however, for a sequence of singular values given by:

$$s_0^N = (5N-4)^2 . \qquad ....(90)$$

At infinity, we seek solutions of the form:

$$\phi(\xi) = a_0\xi^{-4/5} + a_1\xi^{-6/5} + a_2\xi^{-8/5} + ...$$
$$\psi(\xi) = b_0\xi^{-1/5} + b_1\xi^{-3/5} + b_2\xi^{-1} + ... \ . \qquad ....(91)$$

In this series, a_0, b_0 and c_0 can be prescribed arbitrarily and the remaining coefficients can be determined from the equations. We note that the choice of exponents in equation (91) arises naturally by considering the left hand sides in equation (88). The leading terms in (91) make the left hand sides equal to zero and produce terms of lower order on the right hand sides. To compare with Eggers, we note that the Eulerian spatial position x is proportional to the integral of the stretch ϕ, i.e. to $\xi^{1/5}$. The radius is proportional to $\phi^{-1/2}$, i.e. to $\xi^{2/5}$. Hence the radius behaves like the square of the spatial distance and the velocity ψ behaves like the reciprocal of the spatial distance {see equation (14) in Eggers [22]}. The viscous stress is proportional to $\xi^{-2/5}$, i.e. x^{-2}.

We note that the minimum filament radius $\delta s^{-1/2}$ is proportional to $(-t)(max\,\phi)^{-1/2}$. Hence the filament thins linearly in time. The maximum of ϕ is not at $\xi = 0$ (although it is actually fairly close to zero). Consequently, the location of the "pinch point" where the filament has minimum radius moves in space, both in the Lagrangian and Eulerian frame. The velocity with which this point moves is proportional to $(-t)^{\beta-1-\alpha} = (-t)^{-1/2}$.

Since there are two free constants in equation (91) and a general solution of (88) requires three initial conditions, a generic solution of (88) does not have the correct behavior at infinity. Using a shooting method, we can determine a curve $s_0 = f(u_0)$ for the values at zero of those solutions which behave according to (91) as $\xi \to \infty$. Another curve $s_0 = g(u_0)$ yields the values at zero of those solutions which have the correct behavior as $\xi \to \infty$. Because of symmetry considerations, we have $g(u_0) = f(-u_0)$. We then need to find the intersection of these curves, i.e. the value of u_0 for which $g(u_0) = f(u_0)$.

Eggers's similarity solution has an $s_0 = 119.97$ slightly below the third of the singular values ($s_0^3 = 121$), and the corresponding u_0 is $u_0 = 0.785$. There is an infinite sequence of similarity solutions with values of s_0 slightly below s_0^7, s_0^{11}, s_0^{15}, etc. (Brenner [24]). For generic initial data, it has been found that the solution with $N = 3$ is the one relevant for breakup. The Eggers similarity solution is highly asymmetric. Figure 7 shows the jet profile.

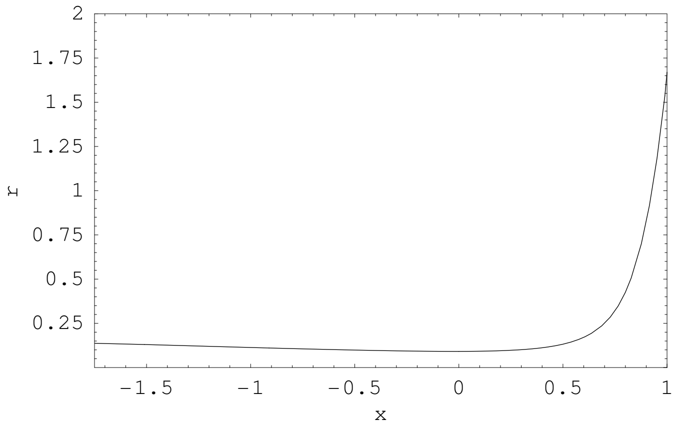

Figure 7: Eggers similarity solution.

5.2 Giesekus fluid

Renardy and Losh [39] have extended the analysis above to a Giesekus fluid with retardation. The retardation parameter $\varepsilon = 1/(\eta\nu)$ is the only dimensionless constant in the equations; here η and ν have the same meaning as in Section 4.2. To find solutions, they employed a continuation method using the Newtonian solution as a starting point, and then gradually increasing ε. They continued this up to $\varepsilon = 20$ and found solutions, which were qualitatively similar to the Newtonian case. The singular values for s_0 shift to:

$$s_0^N = \left[5N - 4 + \frac{2\varepsilon(5N-6)}{5N+6} \right]^2 , \qquad \text{..........(92)}$$

and the value of s_0 for the similarity solution remains slightly below the third of these values. I refer to Renardy [39] for details.

5.3 Generalized Newtonian fluid

For a generalized Newtonian fluid, we have the dimensionless system becomes:

$$u_t = \frac{\partial}{\partial X}\left(\frac{s_t}{s^2}\left|\frac{s_t}{s}\right|^{a-1} + \frac{1}{\sqrt{s}} \right) , \; s_t = u_X . \qquad \text{..........(93)}$$

We seek self-similar solutions of the form:

$$s(X,t) = (-t)^{-\alpha}\phi\left(\frac{X}{(-t)^{\beta}}\right) , \; u(X,t) = (-t)^{-\gamma}\psi\left(\frac{X}{(-t)^{\beta}}\right). \qquad \text{..........(94)}$$

Substitution into equation (93) and equating coefficients of like powers of t results in:

$$\alpha = 2a \; , \; \beta = \frac{3a}{2} + 1 \; , \; \gamma = \frac{a}{2} . \qquad \text{..........(95)}$$

and

$$\gamma\psi + \beta\xi\psi' = \frac{d}{d\xi}\left[\frac{\alpha\phi + \beta\xi\phi'}{\phi^2}\left|\alpha + \beta\xi\frac{\phi'}{\phi}\right|^{a-1} + \phi^{-1/2} \right]. \qquad \text{..........(96)}$$

$$\alpha\phi + \chi\xi\phi' = \psi'$$

When $\beta < \alpha$ ($\alpha > 2$), the reasoning given in Renardy [36] shows that the self-similar region expands rather than shrinks in space as breakup is approached. Such a solution cannot describe breakup in a problem that has boundary conditions imposed at a finite length.

Near $\xi = 0$, we look for analytic solutions with the series expansions:

$$\phi(\xi) = s_0 + s_1\xi + s_2\xi^2 + s_3\xi^3 + \ldots$$
$$\psi(\xi) = u_0 + u_1\xi + u_2\xi^2 + u_3\xi^3 + \ldots \quad \text{..........(97)}$$

As in the Newtonian case, we find that the coefficients s_n and u_n can be determined in terms of s_0 and u_0 , except at singular values of s_0 given by:

$$\sigma_n = (2a)^{2a}\left(n - 2 + \frac{3n}{2}a\right)^2. \quad \text{..........(98)}$$

At $s_0 = \sigma_n$, n = 1, 2, ... , the coefficient s_n becomes infinite and logarithmic terms must be included in the expansion (97).

As $\xi \to \infty$, ϕ and ψ decay to zero. The leading terms near infinity are:

$$\phi(\xi) = p_0\xi^{-\alpha/\beta} + p_1\xi^{(\alpha+1)/\beta} + \ldots$$
$$\psi(\xi) = q_0\xi^{-\gamma/\beta} + q_1\xi^{-(\gamma+1)/\beta} + \ldots, \quad \text{..........(99)}$$

because these make the left hand sides in equation (96) equal to zero and produce terms of lower order on the right hand sides. Substitution in (96) determines p_n and q_n in terms of p_0 and q_0 which are arbitrary.

In [40] a shooting method was used to find similarity solutions. The following results were found:

1. A branch continuing the Eggers solution exists for $0.3 \le a \le 1.97$.
2. Along this branch, asymmetry decreases in both directions as a changes away from the Newtonian value 1.
3. As in the case of no inertia discussed above, the jet profiles become cusped for small a, and U-shaped for large a.
4. Branches of symmetric solutions exist near a =0.3 and near a = 1.97. Along each of these branches, s_0 varies rapidly from σ_2 to σ_4 . At these end-points, the solution becomes equivalent to the inertialess solution after a rescaling of coordinates.
5. The Eggers branch bifurcates from the symmetric solution near a = 0.3. At the upper end, on the other hand, there is a singular limit where s_0 reaches σ_3 and the limiting shape remains asymmetric.

The singular limit points of the symmetric branches provide a link between inertial and inertialess solution, which is not apparent in the Newtonian case.

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